Flat wavelet bases adapted to the homogeneous Sobolev spaces

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Abstract

We present a construction of "flat wavelet bases" adapted to the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$. They solve the problem of the phenomenon of infrared divergence which appears for usual wavelet expansions in $\dot{H}^s(\mathbb{R}^n)$: these bases remove the divergence in the case $s - \frac{n}{2} \notin \mathbb{N}$ since they are also bases of the realization of $\dot{H}^s(\mathbb{R}^n)$. In the critical case $s - \frac{n}{2} \in \mathbb{N}$, they provide a confinement of the divergence in a "small" space.

Key words: Homogeneous Sobolev spaces, realization, wavelet bases, infrared divergence

1 Introduction

The homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ naturally appear in the resolution of problems which possess an invariance by dilations (PDE invariant by scaling, self-similar stochastic processes). These spaces, contrary to their inhomogeneous version, have a dilation invariant norm. But they are only spaces of distributions modulo polynomials and not of distributions. Thus, operations such as multiplication by a function of the Schwarz class or localization are not possible.

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Bourdaud has shown that it was possible to realize the spaces $\dot{H}^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $s - \frac{n}{2} \notin \mathbb{N}$. We can chose, in an homogeneous (the operation commutes with dilation), linear, and continuous way, a representative of each equivalence class, and thus obtain a "realized version" of $\dot{H}^s(\mathbb{R}^n)$ embedded in $\mathcal{S}'(\mathbb{R}^n)$.

If $s < \frac{n}{2}$, the realization is given by the unique representative which belongs to $L^q(\mathbb{R}^n)$ with $\frac{n}{q} = \frac{n}{2} - s$ and an analysis in classical wavelets is perfectly adapted.

If $s > \frac{n}{2}$, the realization is obtained by choosing the representative that vanishes at the origin, as well as its partial derivatives of order less than $E(s - \frac{n}{2})$. Since they do not all vanish at the origin, the classical wavelets are not adapted anymore: the expansion is not necessarily convergent in $\mathcal{S}'(\mathbb{R}^n)$ because of the low-frequency term, giving rise to the phenomenon of infrared divergence.

We propose in this paper the construction of flat wavelets bases. These bases are unconditional bases both of $\dot{H}^s(\mathbb{R}^n)$ and of its realized version. Our method consists of modifying a Daubechies wavelet basis with compact support in order to impose the vanishing conditions at 0. These bases also allow us to study the spaces $\dot{H}^s(\mathbb{R}^n)$ which are not realizable in an homogeneous way ($s - \frac{n}{2} \in \mathbb{N}$). They provide a confinement of the infrared divergence in the sense that we can write $\dot{H}^s(\mathbb{R}^n)$ as the direct sum $X \oplus Y$, where X and Y are two spaces stable by dyadic dilations, Y can be realized, and X is small and regular.

This paper is organized in the following way:

In Section 2, we recall the classical definition of Sobolev spaces and a modified version which allows us to use compact wavelets. For the convenience of the reader we also review known results on realization.

In Section 3, we give a sufficient and necessary condition for a basis of $\dot{H}^s(\mathbb{R}^n)$ to be a basis of the realized space and show that this condition can be satisfied by a basis of wavelets with compact support of $L^2(\mathbb{R}^n)$.

In Sections 4 and 5, we construct the flat wavelet bases and show that they are unconditional bases both of $\dot{H}^s(\mathbb{R}^n)$ and its realization.

In Sections 6, 7 and 8, we study the non-realizable case. In Section 6, we introduce the notion of confinement of infrared divergence; in Section 7, we propose a confinement of the spaces $\dot{H}^s(\mathbb{R}^n)$, $s - \frac{n}{2} \in \mathbb{N}$, and study its properties. In Section 8, we apply this notion to the study of the lifting of the Laplacian in $\dot{H}^2(\mathbb{R}^4)$.

2 Definitions

2.1 Notations

In what follows, $\mathcal{P}$ will denote the subspace of distributions formed by the polynomials and $\mathcal{P}_m$ the subspace formed by the polynomials of degree less than or equal to m ($m \in \mathbb{N}$).

We will denote by $\mathcal{S}_0(\mathbb{R}^n)$ the subspace of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ formed by the functions u satisfying

$$\int x^\alpha u(x)dx = 0 \quad \forall \alpha \in \mathbb{N}^n,$$

and by $\mathcal{S}'_0(\mathbb{R}^n)$ its dual. This space is identified with $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$. For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $[f]$ its class in $\mathcal{S}'_0(\mathbb{R}^n)$.

Finally, for $x \in \mathbb{R}$, $E(x)$ will denote the entire part of x.

2.2 The classical definition

Let ψ be a function that defines a Littlewood-Paley analysis: its Fourier transform $\widehat{\psi}$ belongs to $\mathcal{C}_0^\infty(\mathbb{R}^n)$, is supported on the annulus $\{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\}$ and satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^j \xi) = 1, \quad \xi \neq 0.$$

We define the operator Δ_j, $j \in \mathbb{Z}$, on $\mathcal{S}'(\mathbb{R}^n)$ and on $\mathcal{S}'_0(\mathbb{R}^n)$ by

$$\widehat{(\Delta_j f)} = \widehat{\psi}(2^{-j}\cdot)\widehat{f}. \tag{1}$$

For $s \in \mathbb{R}$, we consider the space $\dot{\mathbb{H}}^s(\mathbb{R}^n)$ consisting of all tempered distributions $f \in \mathcal{S}(\mathbb{R}^n)$ such that

$$R(f) = \left(\sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^2}^2 2^{2js} \right)^{\frac{1}{2}}$$

is finite. The operator R is not a norm (but only a semi-norm) on $\dot{\mathbb{H}}^s(\mathbb{R}^n)$. We have indeed $R(f) = 0$ if and only if f is a polynomial.

Definition 1 *For $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ is the quotient of $\dot{\mathbb{H}}^s(\mathbb{R}^n)$ by $\mathcal{P}$.*

The operator R is now a norm on $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ and $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ is a Banach space of distributions modulo polynomials.

The norm R is homogeneous on $\dot{\mathcal{H}}^s(\mathbb{R}^n)$. That means that for $\lambda > 0$ and $[f] \in \dot{\mathcal{H}}^s(\mathbb{R}^n)$, one has $R([f(\lambda.)]) = \lambda^{s-\frac{n}{2}} R([f])$. This property is one of the main interest to introduce the homogeneous version of the Sobolev spaces. But with that definition, we have to deal with distributions modulo polynomials and not with classical distributions. This difficulty has been studied by Bourdaud in [3] where he has shown in which cases $\dot{H}^s(\mathbb{R}^n)$ can be seen (realized) as a subspace of tempered distribution:

Theorem 2 *The spaces $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $s - \frac{n}{2} \notin \mathbb{N}$, are realizable in a dilation invariant way. In other words, there exists a linear and continuous map $\sigma : \dot{\mathcal{H}}^s(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ such that for all $[u] \in \dot{\mathcal{H}}^s(\mathbb{R}^n)$, one has $[\sigma([u])] = [u]$ (in $\mathcal{S}'_0(\mathbb{R}^n)$) and for all $\lambda > 0$ one has $\sigma([u(\lambda\cdot)]) = (\sigma([u]))(\lambda\cdot)$. The map σ is called the realization of $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ and is unique. The space $\sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n))$ equipped with the norm R and becomes a Banach space.*

If $s - \frac{n}{2} \in \mathbb{N}$, the space $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ is not realizable in a dilation invariant way.

The realization is explicit. If $0 \leq s < \frac{n}{2}$, it is given by the choice of the unique representative u of $[u] \in \dot{\mathcal{H}}^s(\mathbb{R}^n)$ that belongs to $L^q(\mathbb{R}^n)$ with $\frac{n}{q} = \frac{n}{2} - s$.

If $s > \frac{n}{2}$, $s - \frac{n}{2} \notin \mathbb{N}$, the representatives u of $[u] \in \dot{\mathcal{H}}^s(\mathbb{R}^n)$ belongs to $\mathcal{C}^{E(s-\frac{n}{2})}(\mathbb{R}^n)$ and the realization is given by the choice of u satisfying

$$\partial^\alpha u(0) = 0 \quad \text{for all } \alpha \in \mathbb{N}^n,\ |\alpha| \leq E(s - \frac{n}{2}).$$

Now, in the case $s - \frac{n}{2} \notin \mathbb{N}$, the space $\sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n))$ is a subspace of classical distributions. In addition, we have the following properties (cf. [4]).

Proposition 3 *(i) (Hardy inequality) Let $s \geq 0$, $s - \frac{n}{2} \notin \mathbb{N}$. There exists a constant $C > 0$ such that for all function $f \in \sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n))$, one has*

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^{2s}} dx \leq C \|f\|^2_{\dot{\mathcal{H}}^s}.$$

(ii) (Localization) The space $\sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n))$ is localizable. More precisely, for a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, supported in an annulus, and such that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1$, for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$\forall [f] \in \dot{\mathcal{H}}^s(\mathbb{R}^n), \qquad \|[f]\|^2_{\dot{\mathcal{H}}^s} \simeq \sum_{j\in\mathbb{Z}} \|\sigma([f])\varphi_j\|^2_{\dot{\mathcal{H}}^s},$$

where $\varphi_j = \varphi(2^{-j}\cdot)$

Remark The space $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ is defined modulo the polynomials. It only can be analyzed with waves which have infinite vanishing moments. Thus, wavelets bases with compact support can not be bases of this space.

That leads us to the appropriate definition of the Sobolev spaces.

2.3 Modification of the definition

Let us study more precisely the Littlewood-Paley analysis of $f \in \dot{\mathbb{H}}^s(\mathbb{R}^n)$. If $s < \frac{n}{2}$, the Littlewood-Paley expansion of $f \in \dot{\mathbb{H}}^s(\mathbb{R}^n)$ converges in the distributional sense and we have

$$f - \sum_j \Delta_j(f) \in \mathcal{P}.$$

If $s \geq \frac{n}{2}$, the expansion does not necessarely converge in $\mathcal{S}'(\mathbb{R}^n)$. An infrared divergence can appear since the low-frequency term ($j \leq 0$) may diverge. Nevertheless, there exist polynomials P_j of degree less than or equal to $E(s - \frac{n}{2})$, such that the expansion $\sum_{j\in\mathbb{Z}} \Delta_j f - P_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$. That allows us to define the spaces $\dot{\mathbf{H}}^s(\mathbb{R}^n)$, for $s \geq \frac{n}{2}$, by

$$f \in \dot{\mathbf{H}}^s(\mathbb{R}^n) \Leftrightarrow f \in \dot{\mathbb{H}}^s(\mathbb{R}^n) \text{ and there exists polynomials } P_j \text{ of degree less than or equal to } E(s - \frac{n}{2}) \text{ such that } f = \sum_{j\in\mathbb{Z}} \Delta_j f - P_j$$

Now we have $R(f) = 0$ if and only if f is a polynomial of degree less than or equal to $E(s - \frac{n}{2})$.

Definition 4 *For $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is defined by*

$$f \in \dot{H}^s(\mathbb{R}^n) \Leftrightarrow f \in \dot{\mathbb{H}}^s(\mathbb{R}^n) \text{ and } f = \sum_{j\in\mathbb{Z}} \Delta_j f, \qquad \text{if } s < \frac{n}{2}$$

and by the quotient

$$\dot{H}^s(\mathbb{R}^n) = \dot{\mathbf{H}}^s(\mathbb{R}^n)/\mathcal{P}_{E(s-\frac{n}{2})}, \qquad \text{if } s \geq \frac{n}{2}.$$

For $s \geq \frac{n}{2}$ and $f \in \dot{\mathbf{H}}^s(\mathbb{R}^n)$, we denote by $[f]_s$ its class in $\dot{H}^s(\mathbb{R}^n)$.

Remark To analyze the space $\dot{H}^s(\mathbb{R}^n)$, wavelets shall only have $E(s - \frac{n}{2})$ vanishing moments. That allows us to use wavelets with compact support.

Let us recall the following classical properties (with $s \in \mathbb{R}$).

(1) The space $\dot{H}^s(\mathbb{R}^n)$ equipped with the norm R is a Banach space.
(2) The spaces $\dot{H}^s(\mathbb{R}^n)$ and $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ are isometrically isomorphic.
(3) The dual of $\dot{H}^s(\mathbb{R}^n)$ is the space $\dot{H}^{-s}(\mathbb{R}^n)$ for the L^2-scalar product (and $(\dot{\mathcal{H}}^s(\mathbb{R}^n))' = \dot{\mathcal{H}}^{-s}(\mathbb{R}^n)$).
(4) a) If $s \in \mathbb{N}$ and $s < \frac{n}{2}$, the map N defined for $f \in \dot{H}^s(\mathbb{R}^n)$ by

$$N(f) = \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2}$$

is an equivalent norm on $\dot{H}^s(\mathbb{R}^n)$.
b) If $s \in \mathbb{N}$ and $s \geq \frac{n}{2}$, the map N defined for $f \in \dot{\mathbf{H}}^s(\mathbb{R}^n)$ by

$$N(f) = \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2}$$

is a semi-norm on $\dot{\mathcal{H}}^s(\mathbb{R}^n)$ and $N(f) = 0 \Leftrightarrow f \in \mathcal{P}_{E(s-\frac{n}{2})}$. Then, it defines a norm on $\dot{H}^s(\mathbb{R}^n)$.

In what follows we will denote by $\|\cdot\|_{\dot{H}^s}$ the norms R or N (observe that they are equivalent, which is sufficient for our purpose). This norm is homogeneous: for $f \in \dot{H}^s(\mathbb{R}^n)$ and $\lambda > 0$, we obtain $\|f(\lambda\cdot)\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}\|f\|_{\dot{H}^s}$.
(5) a) For $0 \leq s < \frac{n}{2}$, the bilinear form $\langle ., . \rangle$ given, for f and g in $\dot{H}^s(\mathbb{R}^n)$, by

$$\langle f, g \rangle = \sum_{|\alpha|=E(s)} \int\int \frac{(\partial^\alpha f(x) - \partial^\alpha f(y))\overline{(\partial^\alpha g(x) - \partial^\alpha g(y))}}{|x-y|^{n+2(s-E(s))}} dxdy$$

is a scalar product on $\dot{H}^s(\mathbb{R}^n)$.
b) For $s \geq \frac{n}{2}$, for f a representative of $[f]_s \in \dot{H}^s(\mathbb{R}^n)$ and g a representative of $[g]_s \in \dot{H}^s(\mathbb{R}^n)$, the quantity

$$\langle f, g \rangle = \sum_{|\alpha|=E(s)} \int\int \frac{(\partial^\alpha f(x) - \partial^\alpha f(y))\overline{(\partial^\alpha g(x) - \partial^\alpha g(y))}}{|x-y|^{n+2(s-E(s))}} dxdy$$

does not depend of the choice of the representatives and defines a scalar product on $\dot{H}^s(\mathbb{R}^n)$.

Proposition 5 *Let $s \in \mathbb{R}$ be such that $s - \frac{n}{2} \notin \mathbb{N}$. We denote by σ the unique realization of $\dot{\mathcal{H}}(\mathbb{R}^n)$ given in Theorem 2.*

If $s < \frac{n}{2}$, the space $\dot{H}^s(\mathbb{R}^n)$ is already realized and we get

$$\dot{H}^s(\mathbb{R}^n) = \sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n)) \overset{def}{=} \dot{H}^s_{real}(\mathbb{R}^n).$$

If $s \geq \frac{n}{2}$ and $s - \frac{n}{2} \notin \mathbb{N}$, there exists an unique realization $\widetilde{\sigma}$ of $\dot{H}^s(\mathbb{R}^n)$. Moreover, we have

$$\widetilde{\sigma}(\dot{H}^s(\mathbb{R}^n)) = \sigma(\dot{\mathcal{H}}^s(\mathbb{R}^n)) \stackrel{def}{=} \dot{H}^s_{real}(\mathbb{R}^n).$$

It is an immediate consequence of the isometry between $\dot{H}^s(\mathbb{R}^n)$ and $\dot{\mathcal{H}}^s(\mathbb{R}^n)$.

2.4 Identifications

For $s \geq \frac{n}{2}$, our goal is to construct wavelet bases $\{e_k,\ k \in \mathbb{Z}\}$ both of $\dot{H}^s(\mathbb{R}^n)$ and of $\dot{H}^s_{real}(\mathbb{R}^n)$ in the following sense: the functions e_k, $k \in \mathbb{Z}$, form a basis of $\dot{H}^s_{real}(\mathbb{R}^n)$ and their classes $[e_k]_s$, $k \in \mathbb{Z}$, form a basis of $\dot{H}^s(\mathbb{R}^n)$.

In what follows, we will make the identification between an element $[f]_s$ of $\dot{H}^s(\mathbb{R}^n)$ and any of its representatives: we will say that $f \in S'(\mathbb{R}^n)$ belongs to $\dot{H}^s(\mathbb{R}^n)$ if $[f]_s \in \dot{H}^s(\mathbb{R}^n)$ (*i.e.* $f \in \dot{\mathbf{H}}^s(\mathbb{R}^n)$).

Nevertheless, in some cases, we will need to be more careful and make the difference between class and representatives. It will be clearly indicated.

3 Wavelets and realized Sobolev spaces

It is well known that a classical wavelet basis $\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ \varepsilon \in \mathcal{E}\}$ with $\psi^\varepsilon_{j,k} \in \mathcal{C}^r$ (Daubechies or Meyer basis for example) is an unconditional basis of $\dot{H}^s(\mathbb{R}^n)$ for $|s| < r$ (cf [7]). However it is not an unconditional basis of the realized space $\dot{H}^s_{real}(\mathbb{R}^n)$. Let us consider the Meyer basis $\{\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j\cdot - k)\}$ of $L^2(\mathbb{R})$. The wavelet ψ does not vanish at the origin and so does not belong to $\dot{H}^s_{real}(\mathbb{R})$. In fact, an infrared divergence may occur in the wavelet expansion. For example, the series

$$\sum_{j \leq -1} \frac{1}{|j|}\psi(2^j\cdot)$$

converges in $\dot{H}^1(\mathbb{R})$ but for $u \in \mathcal{S}(\mathbb{R})$ with $u \geq 0$ and $\int u = 1$, we have

$$|\langle \sum_{j \leq -1} \frac{1}{|j|}\psi(2^j.), u\rangle| \geq C \sum_{j \leq -1} \frac{1}{|j|}|\psi(0)| \int u = +\infty,$$

and the wavelet expansion does not converge in the distributional sense.

The following criterion gives conditions for a basis of $\dot{H}^s(\mathbb{R}^n)$ to be a basis of $\dot{H}^s_{real}(\mathbb{R}^n)$.

Proposition 6 *Put* $s \geq \frac{n}{2}$, $s - \frac{n}{2} \notin \mathbb{N}$.

1. Let $\{e_k\}_{k\in\mathbb{N}}$ *be a family of tempered distributions such that* $\{e_k\}_{k\in\mathbb{N}}$ *is an unconditional basis of* $\dot{H}^s(\mathbb{R}^n)$. *Then* $\{e_k\}_{k\in\mathbb{N}}$ *is an unconditional basis of* $\dot{H}^s_{real}(\mathbb{R}^n)$ *if and only if, for all* $k \in \mathbb{N}$,

$$e_k \in \dot{H}^s_{real}(\mathbb{R}^n).$$

2. Let $\{f_k\}_{k\in\mathbb{N}}$ *be an unconditional basis of* $\dot{H}^s_{real}(\mathbb{R}^n)$, *orthonormal in* $L^2(\mathbb{R}^n)$. *Then* $\{f_k\}_{k\in\mathbb{N}}$ *is an unconditional basis of* $\dot{H}^s(\mathbb{R}^n)$ *if and only if for all* $k \in \mathbb{N}$,

$$f_k \in \dot{H}^{-s}(\mathbb{R}^n).$$

PROOF. In the following proof, we carefully distinguish the difference between a tempered distribution $f \in \dot{\mathbf{H}}^s(\mathbb{R}^n)$ and its class $[f]_s$ in $\dot{H}^s(\mathbb{R}^n)$.

Remark that only the if part of statement 1. requires any explanation. For the converse, if we explicit the operator of realization σ and if $e_k \in \dot{H}^s_{real}(\mathbb{R}^n)$, then $\sigma([e_k]_s) = e_k$ and thus, by isomorphism, $\{e_k\}_{k\in\mathbb{N}}$ is an unconditional basis of $\dot{H}^s_{real}(\mathbb{R}^n)$.

For 2., if $f_k \in \dot{H}^{-s}(\mathbb{R}^n)$ then, for all $[f]_s \in \dot{H}^s(\mathbb{R}^n)$,

$$g = \sum_{k\in\mathbb{N}} \langle f, f_k\rangle\, f_k$$

is well defined in $\dot{H}^s_{real}(\mathbb{R}^n)$ and $[f]_s = [g]_s$ in $\dot{H}^s(\mathbb{R}^n)$.

Conversely, if $\{[f_k]_s\}_{k\in\mathbb{N}}$ is an unconditional basis of $\dot{H}^s(\mathbb{R}^n)$, for all $[f]_s \in \dot{H}^s(\mathbb{R}^n)$, one has

$$[f]_s = \sum_{k\in\mathbb{N}} c_k([f]_s)[f_k]_s,$$

where the linear form $[f]_s \mapsto c_k([f]_s)$ is continuous. Now, if $[f]_s \in \dot{H}^s(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$, $c_k([f]_s) = \langle f, f_k\rangle$. Thus $f_k \in \dot{H}^{-s}(\mathbb{R}^n)$ by density of $L^2(\mathbb{R}^n) \cap \dot{H}^s(\mathbb{R}^n)$ in $\dot{H}^s(\mathbb{R}^n)$.

Hence, in order to obtain wavelet bases of $\dot{H}^s_{real}(\mathbb{R}^n)$, we need to construct wavelets which vanish at 0 (and their partial derivatives of order less than or equal to $E(s - \frac{n}{2})$). It is impossible to have such a wavelet basis in $L^2(\mathbb{R}^n)$. We have actually the following obstruction.

Theorem 7 *There is no orthonormal basis* $f_\lambda, \lambda \in \Lambda$, *of* $L^2([0, +\infty[)$ *such that*

$$\textit{Supp}\, f_\lambda \subset [a_\lambda, b_\lambda], \quad b_\lambda > a_\lambda > 0, \tag{2}$$

$$\sup_{\lambda\in\Lambda}\frac{b_\lambda}{a_\lambda}<\infty \quad and \qquad \int f_\lambda(t)dt=0 \quad \forall\lambda\in\Lambda. \tag{3}$$

PROOF. Put $C_0=\sup_{\lambda\in\Lambda}\frac{b_\lambda}{a_\lambda}$. One has

$$\mathbf{1}_{[0,1]}=\sum_{\lambda\in\Lambda}\left(\int_0^1\overline{f_\lambda}(x)dx\right)f_\lambda.$$

By (3), if $\lambda\in\Lambda$ is such that $b_\lambda\leq 1$, then $\int_0^1\overline{f_\lambda}(x)dx=0$. Now, by (2), if $b_\lambda\geq 1$ then $a_\lambda\geq\frac{b_\lambda}{C_0}\geq\frac{1}{C_0}$ and we get

$$\mathbf{1}_{[0,1]}=\sum_{\lambda\in\Lambda,\,b_\lambda\geq 1}\left(\int_0^1\overline{f_\lambda}(x)dx\right)f_\lambda.$$

Now we have the required contradiction since the right term is supported on $[\frac{1}{C_0},+\infty$.

The previous theorem can be generalized to the following result.

Theorem 8 *There is no orthonormal basis $e_\lambda,\lambda\in\Lambda=\Lambda_1\cup\Lambda_2$, of $L^2(\mathbb{R}^+)$ such that*

$$\int_{\mathbb{R}^+}e_\lambda(x)dx=0 \quad \forall\lambda\in\Lambda \tag{4}$$

and,

for $\lambda\in\Lambda_1$ *Supp* $e_\lambda\subset[aR_\lambda,bR_\lambda]$ *with* $0<a<b<+\infty$ *and* $R_\lambda>0$,

$$\text{for } \lambda\in\Lambda_2 \quad \text{Supp}\, e_\lambda\subset[0,cR_\lambda] \quad \text{and} \quad |e_\lambda(x)|\leq c_\lambda R_\lambda^{-\frac{1}{2}}\left|\frac{x}{R_\lambda}\right|^\varepsilon$$
$$\text{with} \sup_{l\in\mathbb{Z}}\sum_{\substack{\lambda\in\Lambda_2\\ R_\lambda\simeq 2^l}}c_\lambda^2=M<+\infty.$$

PROOF. Again, we have

$$\mathbf{1}_{[0,1]}=\sum_{\lambda\in\Lambda}\left(\int_0^1\overline{e_\lambda}(x)dx\right)e_\lambda,$$

where the sum for $\lambda\in\Lambda_1$ is supported on $[\frac{a}{b},+\infty[$.

by (4), for $\lambda\in\Lambda_2$, if $cR_\lambda\leq 1$, we have again $\int_0^1\overline{e_\lambda}(x)dx=0$, and, if $cR_\lambda\geq 1$, we get

$$
\begin{aligned}
\left|\sum_{\substack{\lambda\in\Lambda_2\\ cR_\lambda\geq 1}}\left(\int_0^1 \overline{e_\lambda}(x)dx\right)e_\lambda\right| &\leq \sum_{\substack{\lambda\in\Lambda_2\\ cR_\lambda\geq 1}} c_\lambda^2 R_\lambda^{-1-2\varepsilon}|x|^\varepsilon \\
&\leq \sum_{l\in\mathbb{Z}}\sum_{\substack{\lambda\in\Lambda_2\\ cR_\lambda\geq 1\\ R_\lambda\simeq 2^l}} c_\lambda^2 2^{-l(1+2\varepsilon)}|x|^\varepsilon \\
&\leq \sum_{\substack{l\in\mathbb{Z}\\ c2^{l+1}\geq 1}}\left(\sum_{\substack{\lambda\in\Lambda_2\\ R_\lambda\simeq 2^l}} c_\lambda^2\right)2^{-l(1+2\varepsilon)}|x|^\varepsilon \\
&\leq KM|x|^\varepsilon.
\end{aligned}
$$

Finally, we obtain that $\mathbf{1}_{[0,1]}(x) \leq KM|x|^\varepsilon$ for $x \in [0, \frac{a}{b}[$, which give us the required contradiction.

Remark These two last results can be generalized: we can replace $\mathbb{R}^+$ with $\mathbb{R}$ or $\mathbb{R}^n$ (the function $\mathbf{1}_{[0,1]}$ will be replaced with $\mathbf{1}_{[0,1]^n}$ in the proofs). Moreover we do not use in a crucial way the assumption of orthogonality of the basis: we can extend these results to the case of two biorthogonal bases or two dual frames $\{e_\lambda,\, \lambda \in \Lambda\}$ and $\{e^*_\lambda,\, \lambda \in \Lambda\}$ which both satisfy the assumptions of the theorem for the same R_λ.

These results show in particular that there is no Daubechies wavelet basis in which all the wavelets vanish at 0.

4 Reorganized Daubechies wavelet basis

Let us recall our goal. We want to construct a wavelet basis that is an unconditional basis **both** of $\dot{H}^s(\mathbb{R}^n)$ and $\dot{H}^s_{real}(\mathbb{R}^n)$ in the non critical case $s-\frac{n}{2} \notin \mathbb{N}$. For that, we need to keep the property of oscillations (vanishing moments) for the analyzing wavelets and to impose the cancellation at 0 for the synthetizing wavelets. We present a construction which is based on a modification of a Daubechies wavelet basis. The first step is to isolate the wavelets which do not vanish (or their derivatives) at the origin. Let us first recall some properties of the Daubechies wavelet basis (see [6])

Theorem 9 *Let $N \in \mathbb{N}$ be an odd integer. There exists a function $\varphi_N = \varphi$ and $2^n - 1$ functions $\psi^\varepsilon_N = \psi^\varepsilon$, $\varepsilon \in \mathcal{E} = \{1, ..., 2^{n-1}\}$, such that*

(1) $Supp\,\varphi = Supp\,\psi^\varepsilon = [0, N]^n$,

(2) The functions $\varphi(\cdot - k)$, $k \in \mathbb{Z}$, and $2^{\frac{j}{2}}\psi^\varepsilon(2^j \cdot -k)$, $j \geq 0$, $k \in \mathbb{Z}$, $\varepsilon \in \mathcal{E}$, form an unconditional basis of $L^2(\mathbb{R}^n)$,

(3) If $N \geq 3$, φ and ψ^ε belongs to the space $\mathcal{C}^{r(N)}(\mathbb{R}^n)$ where $r(N) > 0$ is a unbounded increasing function of N.

We denote by V_0 the closed subspace of $L^2(\mathbb{R}^n)$ given by

$$V_0 = Span\{\varphi(\cdot - k),\, k \in \mathbb{Z}^n\}$$

and, for $j \in \mathbb{Z}$, we define V_j by the relation

$$f \in V_0 \Leftrightarrow 2^{\frac{nj}{2}} f(2^j \cdot) \in V_j.$$

The sequence $(V_j)_{j\in\mathbb{Z}}$ is the Multiresolution Analysis associated to the wavelet basis. In the same way, we define the closed spaces W_j, $j \in \mathbb{Z}$, by

$$W_0 = Span\{\psi^\varepsilon(\cdot - k),\, k \in \mathbb{Z}^n,\, \varepsilon \in \mathcal{E}\}.$$

and

$$f \in W_0 \Leftrightarrow 2^{\frac{nj}{2}} f(2^j \cdot) \in W_j.$$

Finally, the linear form l_α, for $|\alpha| \leq E(r)$, are defined on $\mathcal{C}^r_{loc}(\mathbb{R}^n)$ by

$$l_\alpha(f) = \partial^\alpha f(0).$$

Proposition 10 *The linear forms l_α, $|\alpha| \leq E(r)$, are linearly independent on W_0.*

The proof will be given at the end of the section.

Let us denote by W_0^0 and W_0^{far} the closed subspaces of W_0 defined by

$$W_0^{far} = Span\{\psi(\cdot + l),\, l \in \mathbb{Z} \setminus \{1, ..., N-1\}\}$$

and

$$W_0^0 = Vect\{\psi(\cdot + l),\, l \in \{1, ..., N-1\}\}. \tag{5}$$

For a function $f \in W_0^{far}$, we have $0 \notin Int(Supp\, f)$ and so $l_\alpha(f) = 0$ for all $|\alpha| \leq E(r)$. It follows from Proposition 10 that the space W_0^0 can be written as $W_{div} \overset{\perp}{\oplus} W_{mod}$ with $W_{mod} = \bigcap_{|\alpha| \leq E(r)} Ker\, l_\alpha$.

We denote by $\{\widetilde{\psi}_k^\varepsilon, (\varepsilon, k) \in \Lambda\}$ an orthonormal basis of W_{mod} and by $\{\tau_\alpha,\, |\alpha| \leq E(r)\}$ a (non-orthogonal) basis of W_{div} such that for all $\beta \in \mathbb{N}^n$, $|\beta| \leq E(r)$

$$\partial^\beta \tau_\alpha = \delta_{\alpha,\beta}.$$

Finally, we denote by $\{\tau_\alpha^*,\, |\alpha| \leq E(r)\}$ the dual basis of $\{\tau_\alpha,\, |\alpha| \leq E(r)\}$ for the L^2-scalar product and we put $\{1, ..., N-1\} \times \mathcal{E} = \Lambda = \Lambda_{mod} \cup \Lambda_{div}$.

The following theorem summarizes the results obtained so far.

Theorem 11 *The union of the three systems*

$$\{\psi^{\varepsilon}_{j,k}(\cdot) = 2^{\frac{nj}{2}} \psi^{\varepsilon}(2^j \cdot - k),\ j \in \mathbb{Z},\ (k, \varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\},$$

$$\{\psi^{\varepsilon}_{j,k}(\cdot) = 2^{\frac{nj}{2}} \psi^{\varepsilon}_k(2^j \cdot),\ j \in \mathbb{Z},\ (k, \varepsilon) \in \Lambda_{mod}\}$$

and

$$\{\tau_{j,\alpha}(\cdot) = 2^{\frac{nj}{2}} \delta_\alpha(2^j \cdot),\ j \in \mathbb{Z},\ |\alpha| \le E(r)\}$$

is an unconditional basis of $L^2(\mathbb{R}^n)$ and of $\dot{H}^s(\mathbb{R}^n)$ for $-r < s < r$, called "reorganized Daubechies basis". The dual basis is the system

$$\{\psi^{\varepsilon}_{j,k},\ j \in \mathbb{Z},\ (k, \varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\} \cup \{\psi^{\varepsilon}_{j,k},\ j \in \mathbb{Z},\ (k, \varepsilon) \in \Lambda_{mod}\}$$
$$\cup \{\tau^*_{j,\alpha} = 2^{\frac{nj}{2}} \tau^*_\alpha(2^j .),\ j \in \mathbb{Z},\ |\alpha| \le E(r)\}.$$

Remark The set Λ_{div} is the set of indices of those wavelets (or one of their partial derivatives) which do not vanish at 0 and lead to the phenomenon of infrared divergence in the wavelet expansion.

Let us now give the proof of Proposition 10. This is a (non-trivial) consequence of the following result (see [1])

Proposition 12 *The linear forms l_α, $|\alpha| \le E(r)$ are linearly independent on V_0.*

PROOF. We study the case of the dimension 1, the argument in dimension n is the same. Let us recall that the scaling function φ and the wavelet ψ are associated to a couple of conjugate mirror filters (m_0, m_1) with

$$m_0(\xi) = \sum_{k=0}^{N} \alpha_k e^{-ik\xi}, \quad \text{with } \alpha_0 \alpha_N \neq 0,$$

$$\frac{1}{2}\varphi(\frac{x}{2}) = \alpha_0 \varphi(x) + ... + \alpha_N \varphi(x - N),$$

$$m_1(\xi) = \sum_{k=0}^{N} \overline{\alpha_{N-k}} e^{-ik\xi},$$

$$\frac{1}{2}\psi(\frac{x}{2}) = \overline{\alpha_N} \varphi(x) + ... + \overline{\alpha_0} \varphi(x - N), \tag{6}$$

and the two relations,

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1 \tag{7}$$

and

$$m_0(\xi)\overline{m_1(\xi)} + m_0(\xi + \pi)\overline{m_1(\xi + \pi)} = 0. \tag{8}$$

It is sufficient to show that the linear forms $l_0, ..., l_m$ defined by

$$\forall i \in \{0, ..., m\} \quad l_i(f) = f^{(i)}(0)$$

are linearly independent on W_0^0 (defined by (5)), with $m = E(r)$. Let $c_0, ..., c_m \in \mathbb{C}$ be such that

$$\sum_{i=0}^{m} c_i l_i = 0. \tag{9}$$

Applying (9) to $\psi(x + l)$, $l \in \{1, ..., N - 1\}$, we get for all $l \in \{1, ..., N - 1\}$,

$$\sum_{i=0}^{m} c_i \psi^{(i)}(l) = 0.$$

Let us put $f = \sum_{i=0}^{m} c_i \psi^{(i)}$, then for all $l \in \mathbb{Z} \setminus \{1, ..., N - 1\}$, $f(l) = 0$. Since $Supp\, f \subset [0, N]$ by (1), f vanishes on $\mathbb{Z}$. But, using (6), we get

$$f(x) = \sum_{k=0}^{N} \overline{\alpha_{N-k}} \sum_{i=0}^{m} c_i 2^{i+1} \varphi^{(i)}(2x - k) = \sum_{k=0}^{N} \overline{\alpha_{N-k}} g(2x - k)$$

where g is defined by

$$g(y) = \sum_{i=0}^{m} c_i 2^{i+1} \varphi^{(i)}(y).$$

It follows that, for all $l \in \mathbb{Z}$,

$$\sum_{k=0}^{N} \overline{\alpha_{N-k}} g(2l - k) = 0. \tag{10}$$

Now, by (1), $Supp\, g \subset [0, N]$, so $g(k) = 0$ for $k \in \mathbb{Z} \setminus \{1, ..., N - 1\}$. Writing $G(\xi) = \sum_{k=1}^{N-1} g(k) e^{-ik\xi}$, the relation (10) is equivalent to

$$m_1(\xi) G(\xi) + m_1(\xi + \pi) G(\xi + \pi) = 0 \quad \forall \xi \in \mathbb{R}.$$

But, m_0 satisfies (7) and (8). Then we have $G(\xi) = \lambda(2\xi)\overline{m_0(\xi)}$ with $\lambda(2\xi) = m_0(\xi)G(\xi) + m_0(\xi + \pi)G(\xi + \pi)$. Hence, $\lambda(2\xi)$ is a trigonometric polynomial of degree less than or equal to N.

We observe that $\overline{m_0(\xi)} = M(e^{i\xi})$ where M is a polynomial which has N complex roots non equal to zero since $\alpha_0 \alpha_N \neq 0$. But $G(\xi) = H(e^{-i\xi})$ where H is a polynomial of degree less than or equal to $N - 1$ and H has N complex roots. Hence, G is identically zero, and g vanishes on $\mathbb{Z}$. That implies that for all $l \in \{1, ..., N - 1\}$,

$$\sum_{i=0}^{m} c_i 2^{i+1} l_i(\varphi(\cdot + l)) = \sum_{i=0}^{m} c_i 2^{i+1} \varphi^{(i)}(l) = 0,$$

and so, according to Proposition 12, $c_0 = ... = c_m = 0$, which ends the proof.

5 Flat wavelet bases and realized Sobolev spaces

We now assume that $r \notin \mathbb{N}$ and fix an integer $m \leq E(r)$. We want to modify the reorganized Daubechies basis to obtain a flat wavelet basis of order m in the following sense.

Definition 13 *A flat wavelet basis of order m (with $m \in \mathbb{N}$) is a r-regular wavelet basis (with $E(r) \geq m$) such that all the wavelets and their partial derivatives of order less than or equal to m vanish at* 0.

In the reorganized Daubechies basis, at the scale $j = 0$, the functions τ_α, for $|\alpha| \leq m$, are the only one which do not satisfy the conditions of cancellation at the origin. We replace them with the functions ω_α ($|\alpha| \leq m$) defined by

$$\omega_\alpha(\cdot) = \tau_\alpha(\cdot) - 2^{|\alpha|}\tau_\alpha(\frac{\cdot}{2}),$$

which now satisfy

$$l_\beta(\omega_\alpha) = 0 \quad \text{for } |\alpha| \leq m \text{ and } |\beta| \leq m.$$

For $j \in \mathbb{Z}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$, we put $\omega_{j,\alpha} = 2^{\frac{jn}{2}}\omega_\alpha(2^j\cdot)$. We have

Proposition 14 *The flat wavelet basis of order m given by*

$$\begin{aligned} &\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\} \cup \{\widetilde{\psi}^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \Lambda_{mod}\} \\ &\cup\{\omega_{j,\alpha},\ j \in \mathbb{Z},\ |\alpha| \leq m\} \cup \{\tau_{j,\alpha},\ j \in \mathbb{Z},\ m+1 \leq |\alpha| \leq E(r)\} \end{aligned}$$

is an unconditional basis of $\dot{H}^s(\mathbb{R}^n)$ for $m + \frac{n}{2} < s < r$ and, for $f \in \dot{H}^s(\mathbb{R}^n)$, we have

$$\begin{aligned} \|f\|^2_{\dot{H}^s} \simeq &\sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\mathbb{Z}^n\times E\setminus\Lambda} |\langle f,\, \psi^\varepsilon_{j,k}\rangle|^2 2^{2js} + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda_{mod}} |\langle f,\, \widetilde{\psi}^\varepsilon_{j,k}\rangle|^2 2^{2js} \\ &+ \sum_{j\in\mathbb{Z}} \sum_{|\alpha|\leq m} |\langle f,\, \omega^*_{j,\alpha}\rangle|^2 2^{2js} + \sum_{j\in\mathbb{Z}} \sum_{m+1\leq|\alpha|E(r)} |\langle f,\, \tau^*_{j,\alpha}\rangle|^2 2^{2js}. \end{aligned}$$

PROOF. We denote by E_α, $|\alpha| \leq m$, and F the closed subspaces of $\dot{H}^s(\mathbb{R}^n)$ defined by

$$E_\alpha = Span\{\tau_{j,\alpha},\ j \in \mathbb{Z}\}$$

and

$$\begin{aligned} F = Span\,(&\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\} \cup \{\widetilde{\psi}^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \Lambda_{mod}\} \\ &\cup\{\tau_{j,\alpha},\ j \in \mathbb{Z},\ m+1 \leq |\alpha| \leq E(r)\}). \end{aligned}$$

We have

$$\dot{H}^s(\mathbb{R}^n) = \underset{|\alpha|\le m}{\oplus} E_\alpha \oplus F.$$

For $|\alpha| \le m$ and for $h = \sum_{|\beta|\le m} e_\beta + f \in \dot{H}^s(\mathbb{R}^n)$ with $e_\beta \in E_\beta$ and $f \in F$, let us define the operators S_α on $\dot{H}^s(\mathbb{R}^n)$ by

$$S_\alpha(h)(\cdot) = 2^{|\alpha|} e_\alpha \left(\frac{\cdot}{2}\right).$$

Finally, let T be defined on $\dot{H}^s(\mathbb{R}^n)$ by

$$T = Id - \sum_{|\alpha|\le m} S_\alpha.$$

Since $m + \frac{n}{2} < s$, one has, for $f \in E_\alpha$ ($|\alpha| \le m$),

$$\|S_\alpha f\|_{\dot{H}^s} = 2^{|\alpha|} 2^{\frac{n}{2}-s} \|f\|_{\dot{H}^s},$$

with $2^{|\alpha|}2^{\frac{n}{2}-s} < 1$. The following lemma allows us to conclude that T is an isomorphism, which ends the proof.

Lemma 15 *Let H be a normed vectorial space and let $E_1, ..., E_n$, F be some vectorial subspaces of H such that*

$$H = E_1 \oplus ... \oplus E_n \oplus F.$$

For $1 \le i \le n$, we consider an endomorphism S_i of H such that
i) $S_i(E_j) = \{0\}$ if $j \ne i$ and $S_i(F) = 0$,
ii)$S_i(E_i) \subset E_i$,
iii) $\forall f \in E_i,\ \|S_i(f)\| \le c\|f\|$ with $c < 1$.
Then the endomorphism T of H defined by

$$T = Id - S_1 - ... - S_n$$

is an isomorphism.

The isomorphism T is explicit. Then we can calculate the dual basis in $\dot{H}^{-s}(\mathbb{R}^n)$. It is the system

$$\begin{gathered}\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\} \cup \{\widetilde{\psi}^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \Lambda_{mod}\} \\ \cup\{\omega^*_{j,\alpha},\ j \in \mathbb{Z},\ |\alpha| \le m\} \cup \{\tau^*_{j,\alpha},\ j \in \mathbb{Z},\ m+1 \le |\alpha| \le E(r)\}.\end{gathered}$$

where $\omega^*_{j,\alpha} = 2^{\frac{nj}{2}} \omega^*_\alpha(2^j\cdot)$ with

$$\omega^*_\alpha = \sum_{p\ge 0} 2^{p(|\alpha|+n)} \tau^*_\alpha(2^p\cdot).$$

Theorem 16 *Let $\frac{n}{2} < s < r$, $s - \frac{n}{2} \notin \mathbb{N}$. The flat wavelet basis of order m of Proposition 14 is an unconditional basis of $\dot{H}^s(\mathbb{R}^n)$ and of $\dot{H}^s_{real}(\mathbb{R}^n)$.*

PROOF. We already know that it is an unconditional basis of $\dot{H}^s(\mathbb{R}^n)$. According to Proposition 6, it is sufficient to show that the functions of this basis belong to $\dot{H}^s_{real}(\mathbb{R}^n)$. That is trivially the case since they and their partial derivatives of order less than or equal to m vanish at 0.

Remark Giving up the L^2-topology, we have succeeded to impose **simultaneously** oscillations and localization (compact support) for the analysing wavelets and cancellations at 0 and localization for the synthesizing wavelets. That allows us an analysis and a synthesis both of $\dot{H}^s(\mathbb{R}^n)$ and $\dot{H}^s_{real}(\mathbb{R}^n)$.

Observe that these properties can not be simultaneously fulfilled in $L^2(\mathbb{R}^n)$ or in the inhomogeneous space $H^s(\mathbb{R}^n)$. The obstruction comes from the fact that in these spaces we have to reconstruct a regular function identically equal to 1 on a neighbourhood of 0 (cf. Theorem 8). This function does not exist anymore in $\dot{H}^s_{real}(\mathbb{R}^n)$ ($s > \frac{n}{2}$) since it does not vanish at 0.

6 Confinement of the infrared divergence

In the case where a subspace E of $\mathcal{S}_0(\mathbb{R}^n)$ is not realizable in a dilation invariant way, we may hope, at least, to confine the divergence in a "small" subspace. This idea is motivated by the following example of the lifting of the Laplacian in $\dot{H}^2(\mathbb{R}^4)$.

We want to solve $\Delta f = g$ with data $g \in L^2(\mathbb{R}^4)$ for $f \in \mathcal{S}'(\mathbb{R}^4)$. The solution is obviously not unique. But assuming that the solution f has partial derivatives of order two in $L^2(\mathbb{R}^4)$ and belongs to $BMO(\mathbb{R}^4)$ reduces the set of solutions of the homogeneous equation is reduced to the constants. The solution f belongs to $\dot{H}^2(\mathbb{R}^4)$ and is unique (in other words, the Laplacian is an isomorphism between $\dot{H}^2(\mathbb{R}^4)$ and $L^2(\mathbb{R}^4)$). Theorem 2 shows that $\dot{H}^2(\mathbb{R}^4)$ is not realizable in a dilation invariant way and so we cannot solve $\Delta f = g$ in $\dot{H}^2(\mathbb{R}^4) \cap \mathcal{S}'(\mathbb{R}^4)$ in a linear, continuous and homogeneous way. However, we can try to reduce the number of degrees of freedom of the obstruction and solve $\Delta f = g$ for $g \in L^2(\mathbb{R}^4)$ in the following form (P):

$$f = T(g) + w$$

with

(1) T a continuous linear operator from $L^2(\mathbb{R}^4)$ to $\mathcal{S}'(\mathbb{R}^4)$ and from $L^2(\mathbb{R}^4)$ to $\dot{H}^2(\mathbb{R}^4)$,

(2) $T(L^2(\mathbb{R}^4))$ a closed subspace of $\dot{H}^2(\mathbb{R}^4)$, stable with respect to dyadic dilation,

(3) $w \in Z$ with Z a closed subspace of $\dot{H}^2(\mathbb{R}^4)$, stable with respect to dyadic dilation,

(4) $\dot{H}^2(\mathbb{R}^4) = T(L^2(\mathbb{R}^4)) \oplus Z$ (direct sum in $\dot{H}^2(\mathbb{R}^4)$, non orthogonal),

(5) For $j \in \mathbb{Z}$, $T(2^{2j} g(2^j \cdot)) = T(g)(2^j \cdot)$.

Remark The choice $T = 0$ and $Z = \dot{H}^2(\mathbb{R}^4)$ is an obvious solution of (P). The goal is to find a solution with a subspace Z, which carries the divergence, as "small" as possible. The word "small" should be understood as a gain of regularity or in the following sense.

Definition 17 *Let E be a subspace of $\mathcal{S}'_0(\mathbb{R}^n)$ (or of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$), equipped with a dyadic dilation invariant Banach structure, such that the canonical injection of E into $\mathcal{S}'_0(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$) is continuous.*

A couple of closed subspaces (X, Y) of E is called a confinement of the infrared divergence of order m if $E = X \oplus Y$ and

(1) X and Y are invariant with respect to dyadic dilation,

(2) Y is realizable in a dyadic dilation invariant way, i.e. there exists a linear and continuous map $\sigma : Y \to \mathcal{S}'(\mathbb{R}^n)$ such that, for all $u \in Y$ and $j \in \mathbb{Z}$, one gets $\sigma(u) = u$ in $\mathcal{S}'_0(\mathbb{R}^n)$ (or in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$) and $\sigma(u(2^j \cdot)) = (\sigma(u))(2^j \cdot)$.

(3) There exist m functions $\theta_1, ..., \theta_m \in E$ such that

$$\{\theta_i(2^j \cdot),\ j \in \mathbb{Z},\ i = 1, ..., m\}$$

is an unconditional basis of X. The space X is called residual space.

The fact that the order m does not depend of the choice of the functions $\theta_1, ..., \theta_m$ is a consequence of the following result.

Theorem 18 *Let B be a Banach space, U an automorphism of B and $n \in \mathbb{N}^*$, such that there exist n vectors $e_1, \ldots, e_n \in B$ for which the collection*

$$\{U^k(e_i)\,;\, k \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$$

is an unconditional basis of B. Let us assume that there exist some vectors $f_j \in B$, indexed by a set E, such that the collection

$$\{U^k(f_j)\,;\, k \in \mathbb{Z}, j \in E\}$$

is also an unconditional basis of B. Then E is finite of cardinality n.

We are now in position to formulate the folloying definition.

Definition 19 *Let B, U and n be as in Theorem 18. The number n is called dimension of B modulo U (or modulo $\mathbb{Z}$).*

PROOF. [of Theorem 18] Observe first that the dual system $\{e^*_{i,k}\}$ (in B') of $\{U^k(e_i)\}$ is of type $\{(U^*)^{-k}(e^*_i)\}$. Indeed, the dual system is completely determined by equations

$$< e^*_{i,k}, U^l(e_j) >= \delta_{i,j}\delta_{k,l},$$

where $< \ , \ >$ denotes the duality B', B. Now, putting $e^*_i = e^*_{i,0}$, we have

$$< (U^*)^{-k}(e^*_i), U^l(e_j) >=< e^*_{i0}, U^{-k+l}(e_j) >= \delta_{i,j}\delta_{0,-k+l} = \delta_{i,j}\delta_{k,l},$$

which proves first assertion. Let us also recall that, for $x \in B$, the series

$$\sum_{i=1}^{n}\sum_{k\in\mathbb{Z}} < (U^*)^{-k}(e^*_i), x > U^k(e_i)$$

converges unconditionaly in B to x and, if $x^* \in B'$, we have

$$< x^*, x >= \sum_{i=1}^{n}\sum_{k\in\mathbb{Z}} < x^*, U^k(e_i) >< (U^*)^{-k}(e^*_i), x >,$$

where the series is absolutely convergent (*i.e.* summable).

These remarks can also be applied to the basis $\{U^k(f_j)\}$ since we do not use the cardinality of E. We denote by $\{(U^*)^{-k}(f^*_j)\}$ its dual system.

Let F be a finite subset of E. Then we have

$$\begin{aligned}\operatorname{Card} F &= \sum_{j\in F} < f^*_j, f_j > \\ &= \sum_{j\in F}\sum_{i=1}^{n}\sum_{k\in\mathbb{Z}} < f^*_j, U^k(e_i) >< (U^*)^{-k}(e^*_i), f_j > \\ &= \sum_{i=1}^{n}\sum_{j\in F}\sum_{k\in\mathbb{Z}} < e^*_i, U^{-k}(f_j) >< (U^*)^{k}(f^*_j), e_i > .\end{aligned}$$

By the previous remarks applied to $\{U^k(f_j)\}$, the family of coefficients

$$< e^*_i, U^{-k}(f_j) >< (U^*)^{k}(f^*_j), e_i >$$

is summable on $\{1, \ldots, n\} \times E \times \mathbb{Z}$ and the sum is

$$\sum_{i=1}^{n}\sum_{j\in E}\sum_{k\in\mathbb{Z}} < e^*_i, U^{-k}(f_j) >< (U^*)^{k}(f^*_j), e_i >= \sum_{i=1}^{n} < e^*_i, e_i >= n.$$

Now, for $\varepsilon > 0$, there exists a finite part F_0 of E such that, for all finite part F of E including F_0, we have $|n - \text{Card}\, F| < \varepsilon$. Taking $\varepsilon = 1/4$, we obtain that E is finite of cardinality n.

7 Confinement of the infrared divergence for non-realizable Sobolev spaces

We now consider the critical case $\dot{H}^s(\mathbb{R}^n)$ with $s - \frac{n}{2} \in \mathbb{N}$. Let us fix s such that $m = s - \frac{n}{2} \in \mathbb{N}$ and choose the regularity r of the wavelets to be greater than s. By extension, the reorganized Daubechies basis will now be called "flat wavelet basis of order -1".

The study of the space is based on the interpolation of the realizable spaces $\dot{H}^{s-\eta}(\mathbb{R}^n)$ and $\dot{H}^{s+\eta}(\mathbb{R}^n)$ for $0 < \eta < min(1, r - s)$. To use the interpolation theory, these spaces have to be embedded in a same space. So, in that section, we embed $\dot{H}^{s-\eta}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}_m$. Again, there exists an unique realization $\widetilde{\sigma}$ of $\dot{H}^{s-\eta}(\mathbb{R}^n)$ and we have

$$\widetilde{\sigma}(\dot{H}^{s-\eta}(\mathbb{R}^n)) = \sigma(\dot{\mathcal{H}}^{s-\eta}(\mathbb{R}^n)) = H^{s-\eta}_{real}(\mathbb{R}^n).$$

For $s - 1 < t < r$, the flat wavelet basis of order $m - 1$ is an unconditional basis of the spaces $\dot{H}^t(\mathbb{R}^n)$. So we can define the closed vectorial subspaces Z^t and Y^t of $\dot{H}^t(\mathbb{R}^n)$ by

$$Z^t = Span\,\{\tau_{j,\alpha},\ j \in \mathbb{Z},\ |\alpha| = m\}$$

and

$$\begin{aligned} Y^t = & Span\{\psi^{\varepsilon}_{j,k},\ j \in \mathbb{Z},\ (k, \varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda\} \cup \{\widetilde{\psi}^{\varepsilon}_{j,k},\ j \in \mathbb{Z},\ (k, \varepsilon) \in \Lambda_{mod}\} \\ & \cup\{\omega_{j,\alpha},\ j \in \mathbb{Z},\ |\alpha| \leq m - 1\} \cup \{\tau_{j,\alpha},\ j \in \mathbb{Z},\ m + 1 \leq |\alpha| \leq E(r)\}. \end{aligned}$$

Theorem 20 *Let s be such that $m = s - \frac{n}{2} \in \mathbb{N}$. The couple of closed subspaces (Z^s, Y^s) of $\dot{H}^s(\mathbb{R}^n)$ is a confinement of the infrared divergence of order $Card\{\alpha \in \mathbb{N}^n;\ |\alpha| = m\}$.*

The basic idea of the proof of the above theorem is to use interpolation theory. We refer to [2] for definitions and results on interpolation methods. Let us fix $0 < \eta < \frac{1}{2}$ such that $s + \eta < r$. We begin to show

Lemma 21 *The spaces Y^s, $Y^{s+\eta}$ and $Y^{s-\eta}$ satisfy*

$$(Y^{s+\eta}, Y^{s-\eta})_{[\frac{1}{2}]} = Y^s. \quad (\textit{Complex interpolation})$$

PROOF. For all $t > 0$ we denote by ℓ^t the space of complex sequences $c = \{c^\varepsilon_{j,k},\ j \in \mathbb{Z},\ (k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda_{div}\} \cup \{c_{j,\alpha},\ j \in \mathbb{Z},\ |\alpha| \leq E(r),\ |\alpha| \neq m\}$ such that

$$\sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda_{div}} |c^\varepsilon_{j,k}|2^{2jt} + \sum_{j\in\mathbb{Z}} \sum_{|\alpha|\leq E(r),\, |\alpha|\neq m} |c_{j,\alpha}|^2 2^{2jt} < +\infty.$$

The operator T defined for $c \in \ell^t$ by

$$\begin{aligned} T(c) = &\sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda} c^\varepsilon_{j,k}\psi^\varepsilon_{j,k} + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda_{mod}} c^\varepsilon_{j,k}\widetilde{\psi}^\varepsilon_{j,k} \\ &+ \sum_{j\in\mathbb{Z}} \sum_{|\alpha|\leq m-1} c_{j,\alpha}\omega_{j,\alpha} + \sum_{j\in\mathbb{Z}} \sum_{m+1\leq|\alpha|\leq E(r)} c_{j,\alpha}\tau_{j,\alpha} \end{aligned}$$

is an isomorphism between ℓ^t and Y^t for $s - \eta \leq t \leq s + \eta$. Hence, it is an isomorphism between $(\ell^{s-\eta}, \ell^{s+\eta})_{[\frac{1}{2}]} = \ell^s$ and $(Y^{s-\eta}, Y^{s+\eta})_{[\frac{1}{2}]}$ and it follows that $(Y^{s-\eta}, Y^{s+\eta})_{[\frac{1}{2}]} = Y^s$.

Now, we can give the proof of Theorem 20

PROOF. For $f \in Y^t$ with $s - 1 < t < r$, we define the operator σ by

$$\begin{aligned} \sigma(f) = &\sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda} \langle f, \psi^\varepsilon_{j,k}\rangle\psi^\varepsilon_{j,k} + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda_{mod}} \langle f, \widetilde{\psi}^\varepsilon_{j,k}\rangle\widetilde{\psi}^\varepsilon_{j,k} \\ &\sum_{j\in\mathbb{Z}} \sum_{|\alpha|\leq m-1} \langle f, \omega^*_{j,\alpha}\rangle\omega_{j,\alpha} + \sum_{j\in\mathbb{Z}} \sum_{m+1\leq|\alpha|\leq E(r)} \langle f, \tau^*_{j,\alpha}\rangle\tau_{j,\alpha}. \end{aligned} \tag{11}$$

We will show that σ is a realization of $Y^{s-\eta}$ and $Y^{s+\eta}$. We distinguish two cases.

First case: $s = \frac{n}{2}$. In that case, we work with the reorganized Daubechies basis and no function $\omega_{j,\alpha}$ appears.

i) σ is a realization of $Y^{s-\eta}$. Indeed, all the wavelets have a compact support. So they belong to $\dot{H}^{s-\eta}_{real}(\mathbb{R}^n)$, since this one is embedded in $L^q(\mathbb{R}^n)$ with $\frac{n}{q} = \frac{n}{2} - (s - \eta)$. Hence, the map σ coincides with the realization of $\dot{H}^{s-\eta}(\mathbb{R}^n)$ on $Y^{s-\eta}$ and then it is a realization of $Y^{s-\eta}$.

ii) σ is a realization of $Y^{s+\eta}$ since all the wavelets which appear in the definition of $Y^{s+\eta}$ vanish at 0 (at the order 0). Then they belong to $\dot{H}^{s+\eta}_{real}(\mathbb{R}^n)$ and σ coincides with the realization of $\dot{H}^{s+\eta}(\mathbb{R}^n)$ on $Y^{s+\eta}$.

Second case: $s \geq \frac{n}{2} + 1$.

i) σ is a realization of $Y^{s-\eta}$. Indeed, the functions $\omega_{j,\alpha}$, $j \in \mathbb{Z}$, $0 \leq |\alpha| \leq m-1$, the functions $\tau_{j,\alpha}$, $j \in \mathbb{Z}$, $m+1 \leq |\alpha| \leq E(r)$, the functions $\widetilde{\psi}^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $(k,\varepsilon) \in \Lambda_{mod}$, and the wavelets $\psi^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $(k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda$, have a compact support and vanish at 0, as well as their partial derivatives of order less than or equal to $m-1 = s-\frac{n}{2}-1$. Again the wavelets belong to $\dot{H}^{s-\eta}_{real}(\mathbb{R}^n)$, and then σ is a realization of $Y^{s-\eta}$.

ii) σ is a realization of $Y^{s+\eta}(\mathbb{R}^n)$ since the functions $\omega_{j,\alpha}$, $j \in \mathbb{Z}$, $|\alpha| \leq m-1$, the functions $\tau_{j,\alpha}$, $j \in \mathbb{Z}$, $m+1 \leq |\alpha| \leq E(r)$, the functions $\widetilde{\psi}^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $(k,\varepsilon) \in \Lambda_{mod}$, and the wavelets $\psi^{\varepsilon}_{j,k}$, $j \in \mathbb{Z}$, $(k,\varepsilon) \in \mathbb{Z}^n \times \mathcal{E} \setminus \Lambda$, have a compact support and vanish at 0, as well as their partial derivatives of order less than or equal to $m = s - \frac{n}{2}$.

By interpolation, we obtain that σ is continuous from Y^s to $\mathcal{S}'(\mathbb{R}^n)$. Then this map is a realization of Y^s and is clearly invariant by dyadic dilations. Moreover, the expansion on the flat wavelet basis of order $s - \frac{n}{2} - 1$ provides automatically the representative of the realized space $\sigma(Y^s)$.

Theorem 22 *Let (Y^s, Z^s) be the confinement given in Theorem 20 and let σ be the realization of Y^s given by the wavelet expansion (formula 11).*

(1) There exists a constant $C > 0$ such that for all $f \in Y^s$,

$$\int \frac{|\sigma(f)(x)|^2}{|x|^{2s}} dx \leq C \|f\|^2_{\dot{H}^s}. \tag{12}$$

(2) The space $\sigma(Y^s)$ is localizable.
(3) If the flat wavelet basis of order $m-1$ is obtained from a Daubechies wavelet basis with a regularity $r \geq s+\frac{n}{2}+1$, then the space Z^s is embedded in the homogeneous Besov space $\dot{B}^{s+\frac{n}{2}}_{1,2}(\mathbb{R}^n)$.

PROOF. The proof of (3) will be given in the next section as a corollary of Theorem 24. Let us now give the proof of the two first assertions.

We fix again $0 < \eta < \frac{1}{2}$ such that $s + \eta < r$. We know that $\sigma(Y^{s-\eta})$ (resp. $\sigma(Y^{s+\eta})$) is embedded in $\dot{H}^{s-\eta}_{real}(\mathbb{R}^n)$ (resp.$\dot{H}^{s+\eta}_{real}(\mathbb{R}^n)$). By Proposition 3, the space $\dot{H}^{s-\eta}_{real}(\mathbb{R}^n)$ (resp.$\dot{H}^{s+\eta}_{real}(\mathbb{R}^n)$) is embedded in $L^2(\mathbb{R}^n, \frac{dx}{|x|^{2(s-\eta)}})$ (resp. $L^2(\mathbb{R}^n, \frac{dx}{|x|^{2(s+\eta)}})$). Hence, σ is a continuous map from $Y^{s-\eta}$ to $L^2(\mathbb{R}^n, \frac{dx}{|x|^{2(s-\eta)}})$ and from $Y^{s+\eta}$ to $L^2(\mathbb{R}^n, \frac{dx}{|x|^{2(s+\eta)}})$. By interpolation, we obtain that σ is continuous from Y^s to $L^2(\mathbb{R}^n, \frac{dx}{|x|^{2s}})$ and we get (12).

Let us now consider a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, with a compact support and such that $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1$, for all $x \in \mathbb{R}^n \setminus \{0\}$. We want to prove that for all $f \in Y^s$,

$$\|f\|^2_{\dot{H}^s} \simeq \sum_{j\in\mathbb{Z}} \|\sigma(f)\varphi_j\|^2_{\dot{H}^s},$$

where $\varphi_j = \varphi(2^{-j}\cdot)$. This property is known for the realized spaces of non critical exponent (cf. Proposition 3). Again we will obtain it by interpolation in the critical case.

For that, we consider, for $t \in \mathbb{R}$, the Hilbert space $\ell^2(\dot{H}^t)$ of the sequences of functions $\{f_l,\, l \in \mathbb{Z}\}$ such that

$$\|\{f_l\}\|_{\ell^2(\dot{H}^t)} := \left(\sum_{l\in\mathbb{Z}} \|f_l\|^2_{\dot{H}^t}\right)^{\frac{1}{2}} \text{ is finite.}$$

Using the complex interpolation method yields for all $t \in \mathbb{R}$ and $\eta \in \mathbb{R}$ (cf. [2])

$$(\ell^2(\dot{H}^{t-\eta}), \ell^2(\dot{H}^{t+\eta}))_{[\frac{1}{2}]} = \ell^2(\dot{H}^t).$$

The linear operator defined for $f \in Y^{s-\eta}$ (resp. $\in Y^{s+\eta}$) by

$$U(f) = \{\sigma(f)\varphi_j,\, j \in \mathbb{Z}\}$$

is continuous from $Y^{s-\eta}$ (resp. $Y^{s+\eta}$) to $\ell^2(\dot{H}^{s-\eta})$ (resp. $\ell^2(\dot{H}^{s-\eta})$). Indeed we know that $\dot{H}^{s-\eta}_{real}(\mathbb{R}^n)$ and $\dot{H}^{s+\eta}_{real}(\mathbb{R}^n)$ are localizable. It follows that

$$\|U(f)\|^2_{\ell^2(\dot{H}^{s-\eta})} = \sum_{j\in\mathbb{Z}} \|\sigma(f)\varphi_j\|^2_{\dot{H}^{s-\eta}} \le K\|f\|^2_{\dot{H}^{s-\eta}}$$

and

$$\|U(f)\|^2_{\ell^2(\dot{H}^{s+\eta})} = \sum_{j\in\mathbb{Z}} \|\sigma(f)\varphi_j\|^2_{\dot{H}^{s+\eta}} \le K\|f\|^2_{\dot{H}^{s+\eta}}.$$

By interpolation, we obtain that U is a continuous operator from Y^s to $\ell^2(\dot{H}^s)$. In particular, one gets, for $f \in Y^s$,

$$\sum_{j\in\mathbb{Z}} \|\sigma(f)\varphi_j\|^2_{\dot{H}^s} \le K\|f\|^2_{\dot{H}^s}.$$

The converse estimation is true on $\dot{H}^s(\mathbb{R}^n)$ and does not depend on the construction of Y^s. We simply choose any representative $f \in \mathcal{S}'(\mathbb{R}^n)$ of $[f]_s \in \dot{H}^s(\mathbb{R}^n)$. So let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $[f]_s \in \dot{H}^s(\mathbb{R}^n)$. For $j \in \mathbb{Z}$, we define $f_j \in \mathcal{S}'(\mathbb{R}^n)$ by $f_j = f\varphi_j$ and we denote by $\Gamma_j = \{x \in \mathbb{R}^n, c_1 2^j \le |x| \le c_2 2^j\}$ the support of f_j. If s is an integer the proof is trivial.

Else, for $t = s - E(s)$ and for the indices $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ with $k - j \ge M$ (where $M \ge 0$ is such that $\Gamma_j \cap \Gamma_k = \emptyset$), we obtain

$$
\begin{aligned}
|\langle f_j,\, f_k\rangle| &= \sum_{|\alpha|=E(s)} \left| \int_{x\in\Gamma_j} \int_{y\in\Gamma_k} \frac{(\partial^\alpha f_j(x)-\partial^\alpha f_j(y))\overline{(\partial^\alpha f_k(x)-\partial^\alpha f_k(y))}}{|x-y|^{n+2r}} dxdy \right| \\
&\le \sum_{|\alpha|=E(s)} \int_{x\in\Gamma_j} \int_{y\in\Gamma_k} \frac{|\partial^\alpha f_j(x)||\overline{\partial^\alpha f_k(y)}|}{2^{k(n+2r)}} dxdy
\end{aligned}
$$

where $\langle .,.\rangle$ is the usual scalar product on $\dot{H}^s(\mathbb{R}^n)$ defined in Section 2. Now,

$$
\begin{aligned}
\int_{x\in\Gamma_j} |\,\partial^\alpha f_j\,(x)|dx \le K &\int_{x\in\Gamma_j} \int_{2c_2 2^j\le|y|\le 4c_2 2^j} |\partial^\alpha f_j(x)-\partial^\alpha f_j(y)| 2^{-nj} dxdy \\
\le\ & K \int_{x\in\Gamma_j} \int_{2c_2 2^j\le|y|\le 4c_2 2^j} \frac{|\partial^\alpha f_j(x)-\partial^\alpha f_j(y)|}{|x-y|^{\frac{n}{2}+r}} 2^{j(\frac{n}{2}+r)} 2^{-nj} dxdy \\
\le\ & K \left(\int_{x\in\Gamma_j} \int_{2c_2 2^j\le|y|\le 4c_2 2^j} \frac{|\partial^\alpha f_j(x)-\partial^\alpha f_j(y)|^2}{|x-y|^{n+2r}} dxdy \right)^{\frac{1}{2}} 2^{j(\frac{n}{2}+r)} \\
\le\ & K 2^{j(\frac{n}{2}+r)} \|f_j\|_{\dot{H}^s}.
\end{aligned}
$$

Then

$$
|\langle f_j,\, f_k\rangle| \le K 2^{(j-k)(\frac{n}{2}+r)} \|f_j\|_{\dot{H}^s} \|f_k\|_{\dot{H}^s}.
$$

Consequently

$$
\begin{aligned}
\|f\|_{\dot{H}^s}^2 &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f_j,\, f_k\rangle \\
&\le 2 \sum_{j\in\mathbb{Z}} \sum_{0\le k-j\le M} |\langle f_j,\, f_k\rangle| + 2 \sum_{j\in\mathbb{Z}} \sum_{k-j\ge M} |\langle f_j,\, f_k\rangle| \\
&\le K \sum_{j\in\mathbb{Z}} \|f_j\|_{\dot{H}^s}^2,
\end{aligned}
$$

which ends the proof of (2).

8 The lifting of the Laplacian in $\dot{H}^2(\mathbb{R}^4)$

In this section, we return to the problem (P) introduced in Section 6, and study the regularity of Z, understood as an embedding in an homogeneous Besov space. But first, we need to prove that the reorganized Daubechies basis and the flat wavelet bases are unconditional basis of the Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ (with conditions on the exponents).

For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the collection of tempered distributions modulo polynomials (or modulo polynomials of degree less than or equal to $E(s - \frac{n}{p})$) f such that

$$f = \sum_{-\infty}^{+\infty} \Delta_j(f) \quad \text{in } \mathcal{S}_0'(\mathbb{R}^n)$$

and

$$\left(2^{js}\|\Delta_j(f)\|_p\right)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}),$$

where Δ_j is defined by (1). Let us recall the following classical result (cf. [7])

Theorem 23 *Let be $1 \leq p, q < \infty$ and $0 \leq s < r$. Let p' (resp. q') be the conjugate exponent of p (resp. q) given by $\frac{1}{p} + \frac{1}{p'} = 1$.*

The Daubechies wavelet basis $\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ \varepsilon \in \mathcal{E}\}$ is an unconditional basis of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and is its own dual system in $\dot{B}^{-s}_{p,q}(\mathbb{R}^n)$. In addition, for $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, one has

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \left(\sum_{j\in\mathbb{Z}} |2^{j(s+\frac{n}{2}-\frac{n}{p})}(\sum_{(k,\varepsilon)\in\mathbb{Z}^n\times E} |\langle f,\, \psi^\varepsilon_{j,k}\rangle|^p)^{\frac{1}{p}}|^q\right)^{\frac{1}{q}}.$$

Moreover, if $p > 1$ and $q > 1$, then the dual system $\{\psi^\varepsilon_{j,k},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n, \varepsilon \in \mathcal{E}\}$ is an unconditional basis of $\dot{B}^{-s}_{p,q}(\mathbb{R}^n)$.

It is obvious that the reorganized Daubechies basis (defined in Theorem 11) is also an unconditional basis of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $1 \leq p, q < \infty$ and $0 \leq s < r$. Indeed the functions τ_α and $\widetilde{\psi}^\varepsilon_k$ are obtained with a change of bases in finite dimension.

Now we consider the flat wavelet basis of order m ($m \in \mathbb{N}$) of Proposition 14. We have

Proposition 24 *The flat wavelet basis of order m is an unconditional basis of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $m + \frac{n}{p} < s < r$, $1 \leq p, q < \infty$, and for $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ we obtain*

$$\|f\|_{\dot{B}^s_{p,q}} \simeq \left(\sum_{j\in\mathbb{Z}} |2^{j(s+\frac{n}{2}-\frac{n}{p})}(\sum_{(k,\varepsilon)\in\mathbb{Z}^n\times E} |\langle f,\, \phi^\varepsilon_{j,k}\rangle|^p)^{\frac{1}{p}}|^q\right)^{\frac{1}{q}},$$

where the functions $\phi^\varepsilon_{j,k}$ for $j \in \mathbb{Z}$ and $(k, \varepsilon) \in \mathbb{Z}^n \times \mathcal{E}$ are a relabelling of the basis. If in addition, one has $p > 1$ and $q > 1$, then the dual system

$$\{\psi^\varepsilon_{j,k},\, j\in\mathbb{Z},\, (k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda\}\cup\{\widetilde{\psi}^\varepsilon_{j,k},\, j\in\mathbb{Z},\, (k,\varepsilon)\in\Lambda_{mod}\}$$
$$\cup\{\omega^*_{j,\alpha},\, j\in\mathbb{Z},\, |\alpha|\le m\}\cup\{\tau^*_{j,\alpha},\, j\in\mathbb{Z},\, m+1\le|\alpha|\le E(r)\},$$

is an unconditional basis of $\dot{B}^{-s}_{p',q'}(\mathbb{R}^n)$.

PROOF. Let us define the closed subspaces E_α for $|\alpha|\le m$, and F of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by

$$E_\alpha = Span\{\tau_{j,\alpha},\, j\in\mathbb{Z}\}$$

and

$$F = Span\left(\{\psi^\varepsilon_{j,k},\, j\in\mathbb{Z},\, (k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda\}\cup\{\widetilde{\psi}^\varepsilon_{j,k},\, j\in\mathbb{Z},\, (k,\varepsilon)\in\Lambda_{mod}\}\right.$$
$$\left.\cup\{\tau_{j,\alpha},\, j\in\mathbb{Z},\, m+1\le|\alpha|\le E(r)\}\right).$$

For $|\alpha|\le m$ and $h=\sum_{|\beta|\le m} e_\beta + f\in\dot{B}^s_{p,q}(\mathbb{R}^n)$ with $e_\beta\in E_\beta$ and $f\in F$, we define on $\dot{B}^s_{p,q}(\mathbb{R}^n)$ the operators

$$S_\alpha(h) = 2^{|\alpha|}e_\alpha(\frac{\cdot}{2}).$$

An easy computation shows that

$$\|S_\alpha(f)\|_{\dot{B}^{s,p}_q} = 2^{|\alpha|}2^{\frac{n}{p}-s}\|f\|_{\dot{B}^{s,p}_q}.$$

If $m+\frac{n}{p}<s$, one has $\|S_\alpha\|<1$. In that case, by Lemma 15, we obtain that the operator $T=Id-\sum_{|\alpha|\le m}S_\alpha$ is an isomorphism which maps the reorganized Daubechies basis on the flat wavelet basis.

The assertion (3) of Theorem 22 follows now obviously from the characterization of wavelet coefficients. Indeed, for $t=s+\frac{n}{2}<r$, the flat wavelet basis of order $m-1$ is an unconditional basis of $\dot{B}^{t,1}_2(\mathbb{R}^n)$. Put $f\in Z\subset\dot{H}^s(\mathbb{R}^n)$. We write

$$f=\sum_{j\in\mathbb{Z}}\sum_{|\alpha|=m} c_{j,\alpha}\tau_{j,\alpha}$$

with

$$\left(\sum_{j\in\mathbb{Z}}\sum_{|\alpha|=m}(|c_{j,\alpha}|2^{js})^2\right)^{\frac{1}{2}}<\infty.$$

But this can be written as

$$\left(\sum_{j\in\mathbb{Z}}\sum_{|\alpha|=m}\left(|c_{j,\alpha}|2^{\frac{jn}{2}}2^{-jn}2^{jt}\right)^2\right)^{\frac{1}{2}}<\infty,$$

and the last series is equivalent to the norm of f in $\dot{B}_q^{t,1}(\mathbb{R}^n)$.

Remark Following the results of Bourdaud in [3] and Youss'fi in [11] on the realization of the homogenous Besov spaces, we can show that the flat wavelet basis of order m is an unconditional basis of the realization of $B_{p,q}^s(\mathbb{R}^n)$ for $m = E(s - \frac{n}{p})$ in the case $s - \frac{n}{p} > 0$, $s - \frac{n}{p} \notin \mathbb{N}$, $1 \leq p, q < \infty$. If $s - \frac{n}{p} \in \mathbb{N}$ and $q > 1$, the homogeneous Besov spaces are not realizable in a dilation invariant way but the flat wavelet basis provides a confinement of the divergence with a gain of regularity for the residual space. In the case $q = p$ we also obtain the localization property and the Hardy inequality.

It is not our purpose here to discuss these results in detail. It can be found in [9], where a generalization to the homogeneous Sobolev spaces $\dot{H}_p^s(\mathbb{R}^n)$ is also given.

8.2 The lifting of the Laplacian

We return to the problem (P) posed in Section 6. The reorganized Daubechies wavelet provides us a solution of (P). Indeed, we have for $f \in \dot{H}^2(\mathbb{R}^4)$ and $g \in L^2(\mathbb{R}^4)$,

$$\begin{aligned}\Delta f = g \Leftrightarrow f &= \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\mathbb{Z}^n\times\mathcal{E}\setminus\Lambda} \langle g,\, \Delta^{-1}\psi_{j,k}^{\varepsilon}\rangle \psi_{j,k}^{\varepsilon} + \sum_{j\in\mathbb{Z}} \sum_{(k,\varepsilon)\in\Lambda_{mod}} \langle g,\, \Delta^{-1}\widetilde{\psi}_{j,k}^{\varepsilon *}\rangle \widetilde{\psi}_{j,k}^{\varepsilon} \\ &+ \sum_{j\in\mathbb{Z}} \sum_{|\alpha|\leq E(r)} \langle g,\, \tau_{j,\alpha}^{*}\rangle \tau_\alpha \\ \Leftrightarrow f &= T(g) + w\end{aligned}$$

where $w = \sum_{j\in\mathbb{Z}} \langle g,\, \tau_{j,0}^{*}\rangle \tau_{j,0}$.

Lemma 25 *The expansion $T(g)$ is convergent in $\mathcal{S}'(\mathbb{R}^n)$.*

PROOF. It is a direct consequence of Theorem 20. The couple (Z, Y) of closed subspaces of $\dot{H}^2(\mathbb{R}^4)$ defined in Section 7 is a confinement of the infrared divergence. In particular Y is realizable and the expansion on the basis provides the realization. Consequently, the series $T(g)$ converges in $\mathcal{S}'(\mathbb{R}^n)$.

That is not the case in general for $w \in Z$ (since $\tau_0(0) = 1$). But by Proposition 22, we know that Z is a subspace of $\dot{B}_{1,2}^4(\mathbb{R}^4)$.

Now, we will show that there is no "better" solution of the problem (P) with the following result.

Theorem 26 *If $(\widetilde{T}, \widetilde{Z})$ is a solution of (P) such that $\widetilde{Z}$ is embedded in a Besov space $\dot{B}^s_{p,q}(\mathbb{R}^4)$ with $s \in \mathbb{R}$, $1 \leq p, q < \infty$, then one has $s = \frac{4}{p}$ and $q \geq 2$. Hence,*

$$\dot{B}^4_{1,2}(\mathbb{R}^4) \subset \dot{B}^s_{p,q}(\mathbb{R}^4).$$

Remark The subspace $\widetilde{Z}$ is dyadically homogeneous of degree 0 (for $w \in \widetilde{Z}$, one has $\|w(2^j \cdot)\|_{\dot{H}^2} = \|w\|_{\dot{H}^2}$). Then it only can be embedded in a Besov space $\dot{B}^s_{p,q}(\mathbb{R}^4)$ with the same homogeneity. Hence $s = \frac{4}{p}$.

For $q \geq 2$, the theorem does not give any information since the Sobolev embeddings give $\dot{B}^4_{1,2}(\mathbb{R}^4) \subset \dot{B}^{\frac{4}{p}}_{p,2}(\mathbb{R}^4) \subset \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4)$. We still have to exclude the case $s = \frac{4}{p}$ and $q < 2$.

PROOF. [of Theorem 26 Assume that there exists $(\widetilde{T}, \widetilde{Z})$ solution of (P) such that $\widetilde{Z}$ is embedded in $\dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4)$ with $1 \leq q < 2$. We will study the structure of $\widetilde{Z}$ to show that it is impossible.

Let us consider the function

$$w = \tau_0 - \widetilde{T}(\tau_0).$$

Clearly, this function belongs to $\widetilde{Z}$. Furthermore, for all sequences $c = \{c_j\} \in \ell^2(\mathbb{Z})$, the expansion $\sum_{j \in \mathbb{Z}} c_j w(2^j.)$ belongs to $\widetilde{Z}$. Indeed, the series is convergent in $\dot{H}^2(\mathbb{R}^4)$ since

$$\begin{aligned} \|\sum_{j \in \mathbb{Z}} c_j w(2^j \cdot)\|_{\dot{H}^2} &= \|(Id - \widetilde{T}\Delta)(\sum_{j \in \mathbb{Z}} c_j \tau_0(2^j \cdot))\|_{\dot{H}^2} \\ &\leq K \|\sum_{j \in \mathbb{Z}} c_j \tau_0(2^j \cdot)\|_{\dot{H}^2} \\ &\leq C \|c\|_{\ell^2} \end{aligned}$$

(one has $\|\sum_{j \in \mathbb{Z}} c_j \tau_0(2^j \cdot)\|_{\dot{H}^2} = \|\sum_{j \in \mathbb{Z}} c_j 2^{-2j} \tau_{0,j}\|_{\dot{H}^2}$). Since $\widetilde{Z}$ is closed in $\dot{H}^2(\mathbb{R}^4)$, the expansion $f = \sum_{j \in \mathbb{Z}} c_j w(2^j \cdot)$ belongs to $\widetilde{Z}$.

Let us now expand w in the reorganized Daubechies basis. There exists a sequence $\alpha = \{\alpha_j\} \in \ell^2$ such that

$$w = \sum_{j \in \mathbb{Z}} \alpha_j \tau_0(2^j \cdot) + v, \tag{13}$$

with $v \in Y$.

Using the assumption that $\widetilde{Z}$ is embedded in $\dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4)$, one has, for all sequence $c = \{c_j\} \in \ell^2(\mathbb{Z})$,

$$\sum_{j\in\mathbb{Z}} c_j w(2^j\cdot) \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4).$$

Taking into account the expansion (13), we arrive at

$$\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \alpha_n c_{j-n}\tau_0(2^j.) + \sum_{j\in\mathbb{Z}} c_j v(2^j.) \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4).$$

Now, by Proposition 24, we have

$$\sum_{j\in\mathbb{Z}} c_j\tau_j + \sum_{j,k} c_{j,k}\psi_{j,k} \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4) \Longrightarrow \sum c_j\tau_j \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4)$$

and

$$\sum c_j\tau_j \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4) \Longleftrightarrow \sum |c_j|^q 2^{jq} < +\infty,$$

or after change of normalization,

$$\sum c_j\tau_0(2^j\cdot) \in \dot{B}^{\frac{4}{p}}_{p,q}(\mathbb{R}^4) \Longleftrightarrow \sum |c_j|^q < +\infty.$$

Finally, we have for all sequence $c = \{c_j\} \in \ell^2(\mathbb{Z})$

$$\alpha * c \in \ell^q(\mathbb{Z})$$

with $q < 2$. The only possibility is then $\alpha = 0$.

But, using the fact that $\alpha = 0$ in (13), we get that

$$\sum_{j<0} \frac{1}{|j|} v(2^j\cdot) = \sum_{j<0} \frac{1}{|j|} w(2^j\cdot) = \sum_{j<0} \frac{1}{|j|}\tau_0(2^j\cdot) - \sum_{j<0} \frac{1}{|j|}\widetilde{T}(\Delta\tau_0)(2^j\cdot) = f_1 - f_2$$

diverges in $\mathcal{S}'(\mathbb{R}^n)$. Indeed, f_1 diverges while f_2 converges by assumption on the space $\widetilde{T}(L^2(\mathbb{R}^4))$. But this series belongs to the realization of Y provided by the reorganized Daubechies basis and is then convergent in $\mathcal{S}'(\mathbb{R}^n)$.

Adapting the proof of this theorem, we can easily show the following result,

Proposition 27 *Let $s \geq 0$, $s - \frac{n}{2} \in \mathbb{N}$. If the couple $(\widetilde{Y}, \widetilde{Z})$ is a confinement of $\dot{H}^s(\mathbb{R}^n)$ such that $\widetilde{Z}$ is embedded in a Besov space $\dot{B}^t_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $1 \leq p, q < \infty$ then we have $t = s - \frac{n}{2} + \frac{n}{p}$ and $q \geq 2$. In particular,*

$$\dot{B}^{s+\frac{n}{2}}_{1,2}(\mathbb{R}^n) \subset \dot{B}^t_{p,q}(\mathbb{R}^n).$$

Thus, the flat wavelet bases provides a confinement with a residual space Z which has a gain of regularity maximal (understood as an embedding in a Besov space).

This paper is a part of the PhD thesis ([9]) and the results -without proofs- have been presented in a Note ([10]).

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